Module-III

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Subtopic: Polar Equation of Conics

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Module 3: Polar Equation of Chords, Tangents and Asymptotes of Conics

1 Equation of Chord of a Conic

To find the equation of the chord joining the points having vectorial angles α and β on the conic $l/r = 1 + e\cos\theta$.

Let PQ be a chord of the conic $l/r = 1 + e\cos\theta$ having extremities P and Q with vectorial angles α and β. Then the polar coordinates of P and Q are $\left(\frac{l}{1+e\cos\alpha}, \alpha\right)$ and $\left(\frac{l}{1+e\cos\beta}, \beta\right)$ respectively. Therefore, the equation of the straight line joining P and Q is given by

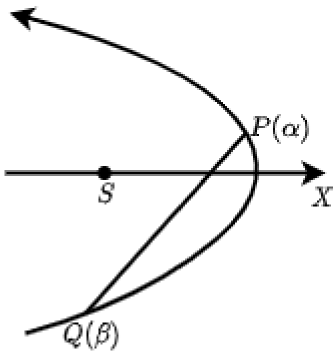

Figure 1: Chord PQ

$$
\begin{aligned}
\frac{\sin(\beta-\alpha)}{r} &= \frac{\sin(\theta-\alpha)}{\frac{l}{1+e\cos\beta}} + \frac{\sin(\beta-\theta)}{\frac{l}{1+e\cos\alpha}} \\
\text{or} \qquad \frac{l}{r}\sin(\beta-\alpha) &= [\sin(\theta-\alpha) + \sin(\beta-\theta)] \\
&\quad + e\,[\cos\beta\sin(\theta-\alpha) + \cos\alpha\sin(\beta-\theta)] \\
&= 2\sin\frac{\beta-\alpha}{2}\cos\frac{2\theta-\alpha-\beta}{2} \\
&\quad + e\,[\cos\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha) + \cos\alpha(\sin\beta\cos\theta - \cos\beta\sin\theta)] \\
&= 2\sin\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos\theta\sin(\beta-\alpha) \\
\textit{i.e.,} \qquad \frac{l}{r} &= \sec\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos\theta. \qquad (1.1)
\end{aligned}
$$

This is the required equation of the chord passing through two points whose vectorial angles are α and β.

Alternative method: Let the equation of the chord PQ be

$$\frac{l}{r} = a\cos\theta + b\sin\theta \tag{i}$$

Since (i) passes through P and Q, we have

$$\frac{l}{SP} = a\cos\alpha + b\sin\alpha \text{ and } \frac{l}{SQ} = a\cos\beta + b\sin\beta. \tag{ii}$$

Also, as P and Q are points on the conic $l/r = 1 + e\cos\theta$, we have

$$\frac{l}{SP} = 1 + e\cos\alpha \text{ and } \frac{l}{SQ} = 1 + e\cos\beta \tag{iii}$$

From (ii) and (iii), we have

$$(a-e)\cos\alpha + b\sin\alpha = 1 \text{ and } (a-e)\cos\beta + b\sin\beta = 1.$$

Solving for $(a-e)$ and b, we get

$$a - e = \frac{\sin\beta - \sin\alpha}{\sin(\beta-\alpha)} = \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\beta-\alpha}{2}} \quad \text{and} \quad b = \frac{\cos\alpha - \cos\beta}{\sin(\beta-\alpha)} = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\beta-\alpha}{2}}.$$

Substituting these values in (i), the equation of the chord is given by

$$\frac{l}{r} = \left[\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\beta-\alpha}{2}} + e\right]\cos\theta + \left[\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\beta-\alpha}{2}}\right]\sin\theta$$

or

$$\frac{l}{r} = \sec\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos\theta$$

Corollary 1: If the extremities of the chord of the conic $l/r = 1 + e\cos\theta$ have vectorial angles $\alpha+\beta$ and $\alpha-\beta$, so that the sum of the angles is 2α and their difference is 2β, then the equation of chord becomes

$$\frac{l}{r} = \sec\beta\cos(\theta-\alpha) + e\cos\theta.$$

Corollary 2: If the equation of the conic is $l/r = 1 + e\cos(\theta-\gamma)$, then the equation of the chord joining two points having vectorial angles α and β is

$$\frac{l}{r} = \sec\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos(\theta-\gamma).$$

2 Equation of the Tangent to a Conic

To find the equation of the tangent to the conic $l/r = 1 + e\cos\theta$ ***at a point having vectorial angle*** α.

Let P be the given point on the conic with vectorial angle α. On the conic take another point Q having vectorial angle β. Then the equation of the chord PQ joining the points $P(\alpha)$ and $Q(\beta)$ is

$$\frac{l}{r} = \sec\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos\theta.$$

Now the tangent at P to the conic is the limiting position of the chord PQ as $Q \to P$, i.e., $\beta \to \alpha$. So taking the limit of the equation of chord PQ as $\beta \to \alpha$, we get the equation of the tangent at P as

$$\frac{l}{r} = \cos(\theta-\alpha) + e\cos\theta. \tag{2.1}$$

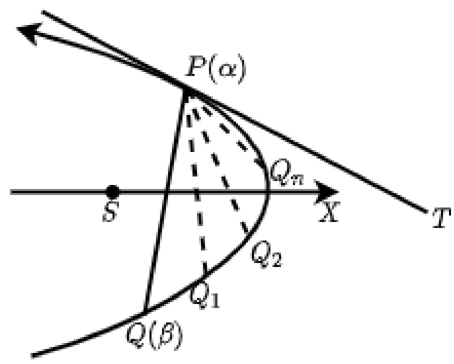

Figure 2: Tangent PT

Corollary 1: If the axis of the conic be inclined at an angle γ to the initial line, so that the equation of the conic is $\frac{l}{r} = 1 + e\cos(\theta-\gamma)$, then the equation of the tangent at the point $'\alpha'$ is obtained by substituting $\alpha-\gamma$ and $\theta-\gamma$ for α and θ in (2.1). Therefore, the equation of tangent at the point $'\alpha'$ is

$$\frac{l}{r} = \cos(\theta-\alpha) + e\cos(\theta-\gamma).$$

Corollary 2: The equation of tangent (2.1) may be written as $\frac{l}{r} = (e+\cos\alpha)\cos\theta + \sin\alpha\sin\theta$, which on transforming in Cartesian coordinates become $(e+\cos\alpha)\,x + \sin\alpha\; y = l$.

$\therefore$ the slope of tangent to the conic at the point $'\alpha'$ is $-\dfrac{(e+\cos\alpha)}{\sin\alpha}$.

2.1 Condition of Tangency

To find the condition so that the line $l/r = a\cos\theta + b\sin\theta$ ***may touch the conic*** $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$***.***

Let the given line touches the given conic at the point $'\alpha'$. Then, its equation must be identical with the equation of the tangent at point $'\alpha'$ to the given conic.
Now, the equation of the tangent at point $'\alpha'$ to the given conic is

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos(\theta - \gamma)$$

or
$$\frac{l}{r} = (\cos\alpha + e\cos\gamma)\cos\theta + (\sin\alpha + e\sin\gamma)\sin\theta$$

Comparing with the equation of given line, we have

$$\cos\alpha + e\cos\gamma = a \quad \text{and} \quad \sin\alpha + e\sin\gamma = b$$

Eliminating α between above equations, we get

$$1 = \cos^2\alpha + \sin^2\alpha = (a - e\cos\gamma)^2 + (b - e\sin\gamma)^2$$

or
$$a^2 + b^2 - 2e(a\cos\gamma + b\sin\gamma) + (e^2 - 1) = 0. \tag{2.2}$$

This is the required condition of tangency.
Corollary: If the equation of the conic is $\frac{l}{r} = 1 + e\cos\theta$, then the condition of tangency is obtained by putting $\gamma = 0$ in (2.2) and becomes

$$(a - e)^2 + b^2 = 1$$

2.2 Point of intersection of tangents

To find the point of intersection of the two tangents at the points $P(\alpha)$ ***and*** $Q(\beta)$ ***on the conic*** $l/r = 1 + e\cos\theta$***.***

The tangents at $P(\alpha)$ and $Q(\beta)$ are given by

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha), \tag{1}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta). \tag{2}$$

Let $R(\rho, \phi)$ be the point of intersection of tangents at P and Q. Then we have

$$\frac{l}{\rho} = e\cos\phi + \cos(\phi - \alpha) \quad \text{and} \quad \frac{l}{\rho} = e\cos\phi + \cos(\phi - \beta) \tag{3}$$

which gives
$$e\cos\phi + \cos(\phi - \alpha) = e\cos\phi + \cos(\phi - \beta)$$

$$\text{or} \quad \cos(\phi - \alpha) = \cos(\phi - \beta) \quad \text{or} \quad (\phi - \alpha) = \pm(\phi - \beta).$$

Since $\alpha \neq \beta$, therefore we get $(\phi - \alpha) = -(\phi - \beta)$ or $\phi = \frac{\alpha+\beta}{2}$.
Putting this value of ϕ in one of the relations in (3), we get

$$\frac{l}{\rho} = e\cos\left(\frac{\alpha + \beta}{2}\right) + \cos\left(\frac{\beta - \alpha}{2}\right)$$

Hence the point of intersection (ρ, ϕ) is given by

$$\phi = \frac{\alpha + \beta}{2} \quad \text{and} \quad \frac{l}{\rho} = e\cos\left(\frac{\alpha + \beta}{2}\right) + \cos\left(\frac{\beta - \alpha}{2}\right).$$

3 Equation to the Asymptotes of a conic

To find the equation of the asymptotes of the conic $l/r = 1 + e\cos\theta$***.***
Let (ρ, α) be a point on the conic $l/r = 1 + e\cos\theta$, then

$$\frac{l}{\rho} = 1 + e\cos\alpha. \tag{1}$$

The equation of the tangent to the conic at the point (ρ, α) is

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta \tag{2}$$

By the definition of asymptote, we know that an asymptote is the limiting position of a tangent as the point of contact tends to infinity. Hence (2) will tend to an asymptote if the point of contact tends to infinity, that is, $\rho \to \infty$.

Now as $\rho \to \infty$, from (1), we have $0 = 1 + e\cos\alpha$, which gives

$$\cos\alpha = -\frac{1}{e} \quad \text{and} \quad \sin\alpha = \pm\frac{\sqrt{e^2-1}}{e}$$

Equation (2) can be written as

$$\frac{l}{r} = (e + \cos\alpha)\ \cos\theta + \sin\alpha\ \sin\theta$$

Putting the values of $\cos\alpha$ and $\sin\alpha$ in above equation, we get

$$\begin{aligned} \frac{l}{r} &= \left(e - \frac{1}{e}\right)\cos\theta \pm \left(\frac{\sqrt{e^2-1}}{e}\right)\sin\theta \\ \text{or} \quad \frac{el}{r} &= (e^2-1)\cos\theta \pm \sqrt{e^2-1}\sin\theta. \end{aligned} \tag{3.1}$$

These are the required equation of the asymptotes to the conic which are real only when $e > 1$.

Solved Examples

Example 1. Let PSQ be a focal chord of the conic $l/r = 1 + e\cos\theta$. Then show that
(a) the tangents at P and Q intersect on the corresponding directrix,
(b) the angle between the tangents at P and Q is $\tan^{-1}\left(\frac{2e\sin\alpha}{1-e^2}\right)$, where α is the angle between chord and the axis of the conic.

Solution. (a) Let the conic be $l/r = 1 + e\cos\theta$. The axis of this conic is the initial line and the focus is pole. Since the focal chord PSQ makes angle α with the axis, the vectorial angles of P and Q are α and $\pi + \alpha$ respectively. Therefore, the equations of the tangents at P and Q are respectively,

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta \tag{1}$$

$$\text{and} \quad \frac{l}{r} = \cos(\theta - \pi - \alpha) + e\cos\theta = -\cos(\theta - \alpha) + e\cos\theta \tag{2}$$

If these tangents intersects in a point $T(\rho, \phi)$, then

$$\frac{l}{\rho} = \cos(\phi - \alpha) + e\cos\phi \quad \text{and} \quad \frac{l}{\rho} = -\cos(\phi - \alpha) + e\cos\phi,$$

which gives $\frac{l}{\rho} = +e\cos\phi$, implying that T lies on the directrix $\frac{l}{r} = +e\cos\theta$.

(b) If m_1 and m_2 be the slopes of the tangents at P and Q, then

$$m_1 = -\frac{e+\cos\alpha}{\sin\alpha} \quad \text{and} \quad m_2 = \frac{e-\cos\alpha}{\sin\alpha}.$$

If ψ is the angle between the tangents at P and Q, then we have

$$\tan\psi = \frac{m_2 - m_1}{1+m_2 m_1} = \frac{\frac{e-\cos\alpha}{\sin\alpha} + \frac{e+\cos\alpha}{\sin\alpha}}{1+\left(\frac{e-\cos\alpha}{\sin\alpha}\right)\cdot\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)} = \frac{2e\sin\alpha}{1-e^2}$$

Therefore, $\psi = \tan^{-1}\left(\frac{2e\sin\alpha}{1-e^2}\right)$.

Example 2. (Auxiliary circle) Prove that the equation to the locus of the foot of perpendicular drawn from focus of the conic $l/r = 1+e\cos\theta$ on a tangent to it is

$$(1-e^2)r^2 + 2elr\cos\theta - l^2 = 0.$$

Solution. The equation of tangent at a point $'\alpha'$ on the given conic is

$$\frac{l}{r} = \cos(\theta-\alpha) + e\cos\theta. \tag{1}$$

The equation of the line perpendicular to this tangent line is given by

$$\frac{L}{r} = \cos\left(\theta + \frac{\pi}{2} - \alpha\right) + e\cos\left(\theta + \frac{\pi}{2}\right)$$

If this line passes through the pole (focus), then $L = 0$. Hence, the equation of the perpendicular from the focus to the tangent is

$$\sin(\theta-\alpha) + e\sin\theta = 0 \tag{2}$$

The foot of perpendicular from the focus to the tangent is obtained by the intersection of (1) and (2). Therefore, the required locus is obtained by eliminating α between (1) and (2), and given by

$$\begin{aligned} \left(\frac{l}{r} - e\cos\theta\right)^2 + (-e\sin\theta)^2 &= \cos^2(\theta-\alpha) + \sin^2(\theta-\alpha) = 1 \\ \text{or} \qquad \frac{l^2}{r^2} - \frac{2le}{r}\cos\theta + e^2 &= 1 \\ \text{or} \qquad (1-e^2)r^2 + 2elr\cos\theta - l^2 &= 0. \end{aligned}$$

This locus represents a circle, when $1-e^2 \neq 0$, that is when the conic is not a parabola. This circle is known as the ***auxiliary circle*** of the conic. When $e = 1$, the locus becomes

$$\frac{l}{r} = 2\cos\theta + \cos(\theta - 0) + \cos\theta,$$

which is the equation of the tangent at the vertex of the parabola.

Example 3. (Director circle) Prove that the locus of the point of intersection of perpendicular tangents of the conic $l/r = 1 + e\cos\theta$ is

$$(1-e^2)r^2 + 2elr\cos\theta - 2l^2 = 0.$$

Solution. The equations of the tangents at the points $'\alpha'$ and $'\beta'$ of the given conic are

$$\frac{l}{r} = \cos(\theta-\alpha) + e\cos\theta \quad \text{and} \quad \frac{l}{r} = \cos(\theta-\beta) + e\cos\theta. \tag{1}$$

If (r', θ') be the point of intersection of two tangents in (1), then

$$\theta' = \frac{\alpha+\beta}{2} \quad \text{and} \quad \frac{l}{r'} = \cos\left(\frac{\alpha-\beta}{2}\right) + e\cos\left(\frac{\alpha+\beta}{2}\right). \tag{2}$$

The slopes of the tangents in (1) are $\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)$ and $\left(-\frac{e+\cos\beta}{\sin\beta}\right)$. Therefore, the tangents in (1) will be perpendicular to each other, if the product of their slopes is -1, that is,

$$\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)\left(-\frac{e+\cos\beta}{\sin\beta}\right) = -1$$

$$\text{or} \quad (e+\cos\alpha)(e+\cos\beta) + \sin\alpha\sin\beta = 0$$

$$\text{or} \quad e^2 + e(\cos\alpha+\cos\beta) + \cos(\alpha-\beta) = 0$$

$$\text{or} \quad e^2 + 2e\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos^2\left(\frac{\alpha-\beta}{2}\right) - 1 = 0$$

Using (2) in above equation,we get

$$e^2 - 1 + 2e\cos\theta'\left(\frac{l}{r'} - e\cos\theta'\right) + 2\left(\frac{l}{r'} - e\cos\theta'\right)^2 = 0$$

$$\text{or} \quad (1-e^2)r'^2 + 2ler'\cos\theta' - 2l^2 = 0$$

Therefore, the locus of (r', θ'), the point of intersection of perpendicular tangents, is

$$(1-e^2)r^2 + 2ler\cos\theta - 2l^2 = 0.$$

When $1-e^2 \neq 0$, that is when the conic is not parabola, this locus represents a circle called the ***director circle*** of the conic. If $e = 1$, the conic is a parabola and the locus reduces to $\frac{l}{r} = \cos\theta$, which is the directrix of the parabola. Hence the locus of the point of intersection of perpendicular tangents to a parabola is its directrix.

Example 4.

Solution.

Example 5. Prove that the portion of the tangent intercepted between the conic and the directrix subtends a right angle at the corresponding focus.

Solution.

Example 6. If PQ is the chord of contact of tangents drawn from a point T to a conic with focus S, then prove that

(i) $SP \cdot SQ = ST^2$, if the conic is a parabola;

(ii) $\frac{1}{SP \cdot SQ} - \frac{1}{ST^2} = \frac{1}{b^2}\sin^2\frac{\angle PSQ}{2}$, if conic is a central conic and b is its semi-minor axis.

Solution.

Example 7. Two equal ellipses of eccentricity e are placed with their axes at right angles and have a common focus S. If PQ is a common tangent to the two ellipses then prove that the $\angle PSQ = 2\sin^{-1}\left(\frac{e}{\sqrt{2}}\right)$.

Solution.

Example 8. Show that the two conics $l_1/r = 1 + e_1\cos\theta$ and $l_2/r = 1 + e_2\cos(\theta-\alpha)$ will touch one another if $l_1^2(1-e_2^2) + l_2^2(1-e_1^2) = 2l_1l_2(1-e_1e_2\cos\alpha)$.

Solution.

Example 9. A chord of a conic subtends a constant angle at a focus of the conic. Show that the chord touches another conic.

Solution.

Example 10. Prove that two points on the conic $l/r = 1 + e\cos\theta$ whose vectorial angles are

α and β respectively will be the extremities of a diameter if $\frac{e+1}{e-1} = \tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2}$.
Solution.
Example 11. A focal chord PSP' of an ellipse is inclined at an angle α to the major axis. Show that the perpendicular from the focus on the tangent at P makes an angle

$$\tan^{-1}\left(\frac{\sin\alpha}{e+\cos\alpha}\right) \text{ with the axis.}$$

Solution.
Example 12. A conic is described having the same focus and eccentricity as the conic $l/r = 1 + e\cos\theta$, and the two conics touch at the point $\theta = \alpha$. Prove that the length of its latus rectum is

$$\frac{2l(1-e^2)}{e^2+2e\cos\alpha+1}, \text{ and that the angle between their axes is } 2\tan^{-1}\left(-\frac{e+\cos\alpha}{\sin\alpha}\right).$$

Solution.
Example 13. If the chord of a conic subtends angle 2α on its focus, then prove that the locus of that point of chord, where bisector of angle 2α meets, is

$$\frac{l\cos\alpha}{r} = 1 + e\cos\alpha\cos\theta.$$

Solution.
Example 14. Prove that the conic (parabola) $\frac{2a}{r} = 1+\cos\theta$ and the conic (parabola) $\frac{2a}{r} = 1-\cos\theta$ intersect orthogonally.
Solution.
Example 15. A tangent drawn from a point P on to the conic $\frac{l}{r} = 1 + e\cos\theta$ make angle β on the focus S. Prove that the locus of mid point of SP is a conic having eccentricity $e\sec\beta$.
Solution.
Example 16. Two conics have a common focus. Then, two of their common chords will pass through the point of intersection of their directrices.
Solution.
Example 17. find the equation of the circle circumscribing the triangle formed by tangents at three given points of parabola $\frac{l}{r} = 1+\cos\theta$.
Solution.